Eq 30:

$$ds^2 = g_{ij} dx^i dx^j$$

$$\Rightarrow \quad \left(\frac{ds}{dt}\right)^2 = g_{ij}\dot{x}^i\dot{x}^j \quad \Rightarrow \quad \frac{ds}{dt} = \left(g_{ij} dx^i dx^j\right)^{1/2} = v$$

Using the chain rule for differentiation, the derivatives of the coordinates with respect to s can now be expressed in terms of their derivatives with respect to t:

Eq 31:

$$\frac{dx^i}{dt} = \frac{dx^i}{ds}\frac{ds}{dt} \quad \Rightarrow \quad \frac{dx^i}{ds} = \frac{1}{v}\frac{dx^i}{dt}$$

Next, use Eq 1 to express derivatives with respect to S in terms of derivatives with respect to t:

Eq 32:

$$dS^2 = 2m(E-V)g_{ij} dx^i dx^j = 2m(E-V)ds^2$$

$$\Rightarrow \quad \frac{dx^i}{dS} = \frac{1}{\sqrt{2m(E-V)}}\frac{dx^i}{ds} = \frac{1}{\sqrt{2m(E-V)}}\frac{1}{v}\frac{dx^i}{dt}$$

Replace v by an equivalent expression in terms of the kinetic energy, T, and use the fact that $E - V = T$. The expression for T at the end of Eq 33 can be found in reference [3] on page 72.

Eq 33:

$$T = \tfrac{1}{2}mv^2 \quad \Rightarrow \quad \frac{1}{v} = \sqrt{\frac{m}{2T}}$$

$$\Rightarrow \quad \frac{dx^i}{dS} = \frac{1}{\sqrt{2mT}}\left(\frac{m}{2T}\right)^{1/2}\dot{x}^i \quad \Rightarrow \quad \left(\frac{dx^i}{dS}\right)_0 = \left[\frac{1}{2T}\dot{x}^i\right]_{t=0}$$

$$\text{where:} \quad T = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)$$

Since $\dot{r}_0$ and $\dot{\theta}_0$ are zero, Eq 33 gives zero for the first two coordinate derivatives. The third coordinate derivative is given by the following equation, where l is the angular momentum:

Eq 34:

$$\left(\frac{dx^2}{dS}\right)_0 = \frac{1}{m\left(r_p^{\,2}\dot{\varphi}_0^{\,2}\right)}\dot{\varphi}_0 = \left[m\left(\frac{l}{m\dot{\varphi}_0}\right)\dot{\varphi}_0^{\,2}\right]^{-1}\dot{\varphi}_0 = \frac{1}{l}$$